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On the derivation of the traces of products of angular momentum operators using operator identities

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Abstract. The products of exponentials containing various angular momentum operators are expressed as a single rotation operator. This gives new expressions for the traces of products of angular momentum operators in terms of hypergeometric functions. Since hypergeometric functions are related to Jacobi polynomials, the traces can also be expressed in terms of these polynomials.

1. Introduction

It is well known that various traces of angular momentum operators are needed in many physical problems such as angular distribution and spin polarisation. It was Rose (1957) who first drew the attention of theorists to this problem. Subsequently Ambler *et al* (1962) provided many of these traces in tabular form. Since the work of Ambler *et al*, many attempts have been made to find simple expressions for such traces. Fairly extensive work in this direction has recently been done by De Meyer and Van den Berghe (1978), who have also given the earlier references to the work done in this area. The purpose of the present work is to provide a new approach to study the trace problem. It is based on the angular momentum operator identities (Ullah 1971). We shall show in the next section that one can express the traces in terms of hypergeometric functions using these operator identities. Concluding remarks will be presented in § 3.

2. Formulation

Let us consider the product of three exponentials containing the operators J_-, J_z, J_+ and write it in terms of a single rotation operator as

$$[\exp(\lambda J_-)][\exp(\theta J_z)][\exp(-\lambda J_+)] = [\exp(-\mathrm{i}\alpha J_z)][\exp(-\mathrm{i}\beta J_y)][\exp(\mathrm{i}\gamma J_z)], \tag{1}$$

where the Euler parameters α, β, γ are related to λ, θ. Since we are interested in the trace of the operators, we rewrite the operator on the right-hand side in expression (1) as

$$[\exp(-\mathrm{i}\alpha J_z)][\exp(-\mathrm{i}\beta J_y)][\exp(-\mathrm{i}\gamma J_z)] = D^{-1}[\exp(-\mathrm{i}\omega J_z)]D, \tag{2}$$

where the operator D is given by

$$D = [\exp(-\mathrm{i}\alpha' J_z)][\exp(-\mathrm{i}\beta' J_y)][\exp(-\mathrm{i}\gamma' J_z)]. \tag{3}$$

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Taking the trace of the operators in expression (1) in the $|jm\rangle$ basis and making use of expression (2), we can write

$$\mathrm{Tr}\{[\exp(\lambda J_-)][\exp(\theta J_z)][\exp(-\lambda J_+)]\} = \sin[\tfrac{1}{2}\omega(2j+1)]/\sin(\tfrac{1}{2}\omega) \tag{4}$$

where ω is related to λ and θ in the following way (Ullah 1971):

$$\cos \tfrac{1}{2}\omega = \tfrac{1}{2}[(1-\lambda^2)\exp(\tfrac{1}{2}\theta) + \exp(-\tfrac{1}{2}\theta)]. \tag{5}$$

Since we are interested (De Meyer and Van den Berghe 1978) in the traces involving various powers of J_+ and J_-, we shall now calculate the coefficient of λ^{2k} in the expansion of the right-hand side of expression (3). Writing $n = 2j+1$ and $\frac{1}{2}\omega = \phi$, the right-hand side of expression (4) becomes $\sin n\phi/\sin\phi$, where $\cos\phi$ is given by

$$\cos\phi = a + b\lambda^2, \tag{6a}$$

$$a = \cosh\tfrac{1}{2}\theta, \qquad b = -\tfrac{1}{2}\exp(\tfrac{1}{2}\theta), \tag{6b}$$

using expression (5). We now write the expansion of $\sin n\phi/\sin\phi$ in terms of powers of $\cos\phi$, using the formula given by Gradshteyn and Ryzhik (1965), as

$$\frac{\sin n\phi}{\sin\phi} = \sum_s (-1)^{s-1}\binom{n-s}{s-1} 2^{n-2s+1}\cos^{n-2s+1}\phi, \tag{7}$$

where $\binom{n-s}{s-1}$ is the binomial coefficient and the sum over s goes from 1 to the integer $< \frac{1}{2}(n+1)$. Using expressions (4), (5), (6) and (7) we obtain

$$\mathrm{Tr}[J_-^k \exp(J_z) J_+^k] = k!\,2^{-k}\exp(\tfrac{1}{2}k\theta)$$

$$\times \sum_s (-1)^{s-1} 2^{n+1-2s} \frac{(n-s)!}{(s-1)!(N-2s)!}\cosh^{N-2s}\tfrac{1}{2}\theta, \tag{8}$$

where $N = n+1-k$.

The summation over s can be carried out in terms of hypergeometric functions (Abramowitz and Stegun 1965, Buchholz 1965) and we finally arrive at the following new result:

$$\mathrm{Tr}[J_-^k \exp(\theta J_z) J_+^k] = \frac{(2j)!k!}{(2j-k)!}\exp(\tfrac{1}{2}k\theta)(2\cosh\tfrac{1}{2}\theta)^{2j-k}$$

$$\times F\left(-\frac{2j-k}{2}, -\frac{2j-k-1}{2}; -2j; \operatorname{sech}^2\tfrac{1}{2}\theta\right). \tag{9}$$

We shall now study special cases of expression (9). Putting $\theta = 0$, we get

$$\mathrm{Tr}(J_-^k J_+^k) = \frac{(2j)!k!}{(2j-k)!}2^{2j-k}F\left(-\frac{2j-k}{2}, -\frac{2j-k-1}{2}; -2j: 1\right). \tag{10}$$

Putting in the value of the hypergeometric function in the above expression in terms of factorials (Abramowitz and Stegun 1965, Buchholz 1965), we have

$$\mathrm{Tr}(J_-^k J_+^k) = \frac{(k!)^2(2j+k+1)!}{(2k+1)!(2j-k)!}. \tag{11}$$

In this form the above result was obtained earlier by using the Wigner–Eckart theorem and also by carrying out the summation over binomial coefficients (De Meyer and Van den Berghe 1978, Rashid 1979).

Next we evaluate the trace of $J_-^k J_z^p J_+^k$ by equating the coefficient of θ^p in expression (9). This gives us

$$\operatorname{Tr}(J_-^k J_z^p J_+^k) = p!c_p, \tag{12}$$

where c_p is the coefficient of θ^p in the expansion of the function $g(\theta)$ given by

$$g(\theta) = \frac{(2j)!k!}{(2j-k)!} \exp(\tfrac{1}{2}k\theta)(2\cosh\tfrac{1}{2}\theta)^{2j-k} F\left(-\frac{2j-k}{2}, -\frac{2j-k-1}{2}; -2j; \operatorname{sech}^2\tfrac{1}{2}\theta\right). \tag{13}$$

Using the properties of the hypergeometric functions, the function $g(\theta)$ can be rewritten as†

$$g(\theta) = \frac{(2j)!k!}{(2j-k)!} \exp(j\theta) F(-2j+k, 1+k; -2j; \exp(-\theta)), \tag{14a}$$

or

$$g(\theta) = \sum_{n=0}^{2j-k} \frac{\Gamma(2j+1-n)\Gamma(k+1+n)}{\Gamma(2j-k+1-n)\Gamma(1+n)} \exp(j-n)\theta. \tag{14b}$$

Differentiating expression (14a) with respect to θ, we obtain the following recursive relation for the coefficients c_p:

$$\begin{aligned}(p+1)(p+2k+2)c_{p+1}& \\ &= (q+\tfrac{3}{4}k^2+\tfrac{1}{2}k)c_p - \sum_{n=2}^{p} \frac{n(n-1)}{(p+2-n)!}c_n - \sum_{n=1}^{p} \frac{n}{(p+1-n)!}c_n \\ &\quad -(q-\tfrac{1}{2}k-\tfrac{1}{4}k^2)\sum_{n=0}^{p}\frac{1}{(p-n)!}c_n, \qquad p=1,2,\ldots,\end{aligned} \tag{15}$$

and

$$c_0 = \frac{(k!)^2(2j+k+1)!}{(2k+1)!(2j-k)!}, \qquad c_1 = \tfrac{1}{2}kc_0 \qquad \text{and} \qquad q = \tfrac{1}{4}(-2j+k)(2j+k+2).$$

Comparing this derivation with the one given by De Meyer and Van den Berghe (1978) in their second paper, we find that the use of hypergeometric functions has considerably simplified the procedure of expressing the higher traces of $(J_-^k J_z^p J_+^k)$ in terms of the lower traces. We also remark here that expression (9) can also be used to establish the consistency relation of De Meyer and Van den Berghe (1978) given by equation (3.3) in their second paper.

3. Concluding remarks

We have shown that angular momentum operator identities can be used to obtain expressions for the traces of the products of angular momentum operators in terms of hypergeometric functions. Since hypergeometric functions are also related to Jacobi

† Relations (14) are due to the referee of this paper. The author would like to thank him for pointing out these simple relations.

polynomials (Abramowitz and Stegun 1965) the traces can also be expressed in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ or $G_n(p, q, x)$. As shown in the last section considerable simplification arises in expressing the trace of $J_-^k J_z^p J_+^k$ in terms of the lower traces, using the present procedure in place of the direct action with $J_\pm$, J_z operators.

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